

YANG-MILLS-HIGGS CONNECTIONS ON CALABI-YAU MANIFOLDS

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ABSTRACT. Let X be a compact connected Kähler–Einstein manifold with $c_1(TX) \geq 0$. If there is a semistable Higgs vector bundle (E, θ) on X with $\theta \neq 0$, then we show that $c_1(TX) = 0$; any X satisfying this condition is called a Calabi–Yau manifold, and it admits a Ricci–flat Kähler form [Ya]. Let (E, θ) be a polystable Higgs vector bundle on a compact Ricci–flat Kähler manifold X . Let h be an Hermitian structure on E satisfying the Yang–Mills–Higgs equation for (E, θ) . We prove that h also satisfies the Yang–Mills–Higgs equation for $(E, 0)$. A similar result is proved for Hermitian structures on principal Higgs bundles on X satisfying the Yang–Mills–Higgs equation.

1. INTRODUCTION

Let X be a compact connected Kähler–Einstein manifold with $c_1(TX) \geq 0$. A Higgs vector bundle on X is a holomorphic vector bundle E on X equipped with a holomorphic section θ of $\text{End}(E) \otimes \Omega_X$ such that $\theta \wedge \theta = 0$. The definition of semistable and polystable Higgs vector bundles is recalled in Section 2. We prove that if there is a semistable Higgs vector bundle (E, θ) on X with $\theta \neq 0$, then $c_1(TX) = 0$ (see Proposition 2.1).

Let X be a compact connected Calabi–Yau manifold, which means that X is a Kähler manifold with $c_1(TX) = 0$. Fix a Ricci–flat Kähler form on X [Ya]. Let (E, θ) be a polystable Higgs vector bundle on X . Then there is a Hermitian structure on E that satisfies the Yang–Mills–Higgs equation for (E, θ) (this equation is recalled in Section 2). Fix a Hermitian structure h on E satisfying the Yang–Mills–Higgs equation for (E, θ) .

Our main theorem (Theorem 3.3) says that h also satisfies the Yang–Mills–Higgs equation for $(E, 0)$.

We give an example to show that if a Hermitian structure h_0 on E satisfies the Yang–Mills–Higgs equation for $(E, 0)$, then h_0 does not satisfy the Yang–Mills–Higgs equation for a general polystable Higgs vector bundle of the form (E, θ) (see Remark 3.4). In

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Remark 3.5 we describe how a Yang–Mills–Higgs Hermitian structure for (E, θ) can be constructed from a Yang–Mills–Higgs Hermitian structure for $(E, 0)$.

Theorem 3.3 extends to the more general context of principal G –bundles on X with a Higgs structure, where G is a connected reductive affine algebraic group defined over \mathbb{C} ; this is carried out in Section 4.

2. HIGGS FIELD ON A KÄHLER–EINSTEIN MANIFOLD

We recall that a Kähler metric is called *Kähler–Einstein* if its Ricci curvature is a constant real multiple of the Kähler form. Let X be a compact connected Kähler manifold admitting a Kähler–Einstein metric. We assume that $c_1(TX) \geq 0$; this is equivalent to the condition that the above mentioned scalar factor is nonnegative. Fix a Kähler–Einstein form ω on X . The cohomology class in $H^2(X, \mathbb{R})$ given by ω will be denoted by $\tilde{\omega}$.

Define the *degree* of a torsionfree coherent analytic sheaf F on X to be

$$\text{degree}(F) := (c_1(F) \cup \tilde{\omega}^{d-1}) \cap [X] \in \mathbb{R},$$

where d is the complex dimension of X . Throughout this paper, stability will be with respect to this definition of degree.

The holomorphic cotangent bundle of X will be denoted by Ω_X . A *Higgs field* on a holomorphic vector bundle E on X is a holomorphic section θ of $\text{End}(E) \otimes \Omega_X = (E \otimes \Omega_X) \otimes E^*$ such that

$$\theta \wedge \theta = 0. \quad (2.1)$$

A *Higgs vector bundle* on X is a pair of the form (E, θ) , where E is a holomorphic vector bundle on X and θ is a Higgs field on E .

A Higgs vector bundle (E, θ) is called *stable* (respectively, *semistable*) if for all nonzero coherent analytic subsheaves $F \subset E$ with $0 < \text{rank}(F) < \text{rank}(E)$ and $\theta(F) \subseteq F \otimes \Omega_X$, we have

$$\frac{\text{degree}(F)}{\text{rank}(F)} < \frac{\text{degree}(E)}{\text{rank}(E)} \quad (\text{respectively, } \frac{\text{degree}(F)}{\text{rank}(F)} \leq \frac{\text{degree}(E)}{\text{rank}(E)}).$$

A semistable Higgs vector bundle (E, θ) is called *polystable* if it is a direct sum of stable Higgs vector bundles.

Let Λ_ω denote the adjoint of multiplication of differential forms on X by ω . In particular, Λ_ω sends a (p, q) –form on X to a $(p-1, q-1)$ –form. Given a Higgs vector bundle (E, θ) on X , the *Yang–Mills–Higgs* equation for the Hermitian structures h on E states that

$$\Lambda_\omega(\mathcal{K}_h + \theta \wedge \theta^*) = c\sqrt{-1} \cdot \text{Id}_E, \quad (2.2)$$

where $\mathcal{K}_h \in C^\infty(X, \text{End}(E) \otimes \Omega_X^{1,1})$ is the curvature of the Chern connection on E for h , the adjoint θ^* of θ is with respect to h , and c is a constant scalar (it lies in \mathbb{R}). A Hermitian structure on E is called Yang–Mills–Higgs for (E, θ) if it satisfies the equation in (2.2).

Proposition 2.1. *If there is a semistable Higgs bundle (E, θ) on X such that $\theta \neq 0$, then $c_1(TX) = 0$.*

Proof. The Higgs field θ on E induces a Higgs field on $\text{End}(E)$, which we will denote by $\widehat{\theta}$. We recall that for any locally defined holomorphic sections s of $\text{End}(E)$,

$$\widehat{\theta}(s) = [\theta, s].$$

Let

$$\theta' = \widehat{\theta} \otimes \text{Id}_{\Omega_X}. \quad (2.3)$$

This is a Higgs field for $\text{End}(E) \otimes \Omega_X$. We note that the integrability condition in (2.1) implies that $\theta'(\theta) = 0$.

Assume that (E, θ) is semistable with $\theta \neq 0$, and also assume that $c_1(TX) \neq 0$. Since (X, ω) is Kähler–Einstein with $c_1(TX) \geq 0$, the condition $c_1(TX) \neq 0$ implies that the anti-canonical line bundle $\bigwedge^d TX$ is positive, so X is a complex projective manifold. Also, the cohomology class of ω is a positive multiple of the ample class $c_1(TX)$.

We shall use the fact that the tensor product of semistable Higgs bundles on a polarized complex projective manifold, with the induced Higgs field, is semistable [Si2, Cor. 3.8]. Thus, $(\text{End}(E), \widehat{\theta})$ is semistable. Moreover, since ω is Kähler–Einstein, Ω_X is a polystable vector bundle, in particular it is semistable. Then $(\Omega_X, 0)$ is a semistable Higgs bundle. As a result, the Higgs bundle $(\text{End}(E) \otimes \Omega_X, \theta')$ is semistable.

The homomorphism

$$\mathcal{O}_X \longrightarrow \text{End}(E) \otimes \Omega_X, \quad f \longmapsto f\theta$$

defines a homomorphism of Higgs vector bundles

$$\varphi : (\mathcal{O}_X, 0) \longrightarrow (\text{End}(E) \otimes \Omega_X, \theta'). \quad (2.4)$$

As $\theta \neq 0$, the homomorphism φ in (2.4) is nonzero. Since $(\text{End}(E) \otimes \Omega_X, \theta')$ is semistable, we have

$$0 = \frac{\text{degree}(\mathcal{O}_X)}{\text{rank}(\mathcal{O}_X)} = \frac{\text{degree}(\varphi(\mathcal{O}_X))}{\text{rank}(\varphi(\mathcal{O}_X))} \leq \frac{\text{degree}(\text{End}(E) \otimes \Omega_X)}{\text{rank}(\text{End}(E) \otimes \Omega_X)} = \frac{\text{degree}(\Omega_X)}{\text{rank}(\Omega_X)}; \quad (2.5)$$

the last equality follows from the fact that $c_1(\text{End}(E)) = 0$. Therefore,

$$\text{degree}(\Omega_X) \geq 0. \quad (2.6)$$

Recall that $c_1(TX) \geq 0$ and X admits a Kähler–Einstein metric. So, (2.6) contradicts the assumption that $c_1(TX) \neq 0$. Therefore, we conclude that

$$c_1(TX) = 0. \quad (2.7)$$

Consequently, ω is Ricci–flat, in particular, X is a Calabi–Yau manifold. \square

A well-known theorem due to Simpson says that E admits an Hermitian structure that satisfies the Yang–Mills–Higgs equation for (E, θ) if and only if (E, θ) is polystable [Si1, Thm. 1] (see also [Si2]); when X is a compact Riemann surface and $\text{rank}(E) = 2$, this was first proved in [Hi].

The Chern connection on E for h will be denoted by ∇^h . Let $\widehat{\nabla}^h$ denote the connection on $\text{End}(E) = E \otimes E^*$ induced by ∇^h . The Levi–Civita connection on Ω_X associated to ω and the connection $\widehat{\nabla}^h$ on $\text{End}(E)$ together produce a connection on $\text{End}(E) \otimes \Omega_X$. This connection on $\text{End}(E) \otimes \Omega_X$ will be denoted by $\nabla^{\omega, h}$.

Proposition 2.2. *Assume that the Hermitian structure h satisfies the Yang–Mills–Higgs equation in (2.2) for (E, θ) . Then the section θ of $\text{End}(E) \otimes \Omega_X$ is flat (meaning covariantly constant) with respect to the connection $\nabla^{\omega, h}$ constructed above.*

Proof. The Hermitian structure h on E produces an Hermitian structure on $\text{End}(E)$, which will be denoted by \widehat{h} . The connection $\widehat{\nabla}^h$ on $\text{End}(E)$ defined earlier is in fact the Chern connection for \widehat{h} . The Kähler form ω and the Hermitian structure \widehat{h} together produce an Hermitian structure on $\text{End}(E) \otimes \Omega_X$. This Hermitian structure on $\text{End}(E) \otimes \Omega_X$ will be denoted by h^ω . We note that the connection $\nabla^{\omega, h}$ in the statement of the proposition is the Chern connection for h^ω .

Since ω is Kähler–Einstein, the Hermitian structure on Ω_X induced by ω satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle $(\Omega_X, 0)$. As h satisfies the Yang–Mills–Higgs equation for (E, θ) , this implies that h^ω satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle $(\text{End}(E) \otimes \Omega_X, \theta')$ constructed in (2.3). In particular, the Higgs vector bundle $(\text{End}(E) \otimes \Omega_X, \theta')$ is polystable. The Proposition is obvious if $\theta = 0$. Assume that $\theta \neq 0$; then φ defined in (2.4) is nonzero.

Since $c_1(\Omega_X) = 0$, the inequality in (2.5) is an equality. Now from [Si1, Prop. 3.3] it follows immediately that

- $\varphi(\mathcal{O}_X)$ in (2.4) is a subbundle of $\text{End}(E)$,
- the orthogonal complement $\varphi(\mathcal{O}_X)^\perp \subset \text{End}(E) \otimes \Omega_X$ of $\varphi(\mathcal{O}_X)$ with respect to the Yang–Mills–Higgs Hermitian structure h^ω is preserved by θ' , and
- $(\varphi(\mathcal{O}_X)^\perp, \theta'|_{\varphi(\mathcal{O}_X)^\perp})$ is polystable with

$$\frac{\text{degree}(\varphi(\mathcal{O}_X)^\perp)}{\text{rank}(\varphi(\mathcal{O}_X)^\perp)} = \frac{\text{degree}(\text{End}(E) \otimes \Omega_X)}{\text{rank}(\text{End}(E) \otimes \Omega_X)} = 0.$$

We note that [Si1, Prop. 3.3] also says that the Hermitian structure on the image of φ induced by h^ω satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle $(\varphi(\mathcal{O}_X), 0)$. Since the above orthogonal complement $\varphi(\mathcal{O}_X)^\perp \subset \text{End}(E) \otimes \Omega_X$ is a holomorphic subbundle,

- the connection $\nabla^{\omega, h}$ preserves $\varphi(\mathcal{O}_X)$,
- and the connection on $\varphi(\mathcal{O}_X)$ obtained by restricting $\nabla^{\omega, h}$ coincides with the Chern connection for the Hermitian structure $h^\omega|_{\varphi(\mathcal{O}_X)}$.

Also, recall that $h^\omega|_{\varphi(\mathcal{O}_X)}$ satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle $(\varphi(\mathcal{O}_X), 0)$. These together imply that all holomorphic sections of $\varphi(\mathcal{O}_X)$ over X are flat with respect to the Yang–Mills–Higgs connection $\nabla^{\omega, h}$ on $\text{End}(E) \otimes \Omega_X$. In particular, the section θ is flat with respect to $\nabla^{\omega, h}$. \square

2.1. Decomposition of a Higgs field. In view of Proposition 2.1, henceforth we assume that $c_1(TX) = 0$. Therefore, the Kähler–Einstein form ω is Ricci–flat. For any point $x \in X$, the fiber of the vector bundle Ω_X over x will be denoted by $\Omega_{X,x}$.

Let (E, θ) be a polystable Higgs vector bundle on X . For any point $x \in X$, we have a homomorphism

$$\eta_x : T_x X \longrightarrow \text{End}(E_x), \quad \eta_x(v) = i_v(\theta(x)), \quad (2.8)$$

where $i_v : \Omega_{X,x} \longrightarrow \mathbb{C}$, $z \longmapsto z(v)$, is the contraction of forms by the tangent vector v .

Lemma 2.3. *For any two points x and y of X , there are isomorphisms*

$$\alpha : T_x X \longrightarrow T_y X \quad \text{and} \quad \beta : E_x \longrightarrow E_y$$

such that $\beta(\eta_x(v)(u)) = (\eta_y(\alpha(v)))(\beta(u))$ for all $v \in T_x X$ and $u \in E_x$.

Proof. Let h be an Hermitian structure on E satisfying the Yang–Mills–Higgs equation for (E, θ) . As before, the Chern connection on E associated to h will be denoted by ∇^h .

Fix a C^∞ path $\gamma : [0, 1] \longrightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Take α to be the parallel transport of $T_x X$ along γ for the Levi–Civita connection associated to ω . Take β to be the parallel transport of E_x along γ for the above connection ∇^h . Using Proposition 2.2 it is straightforward to deduce that

$$\beta(\eta_x(v)(u)) = (\eta_y(\alpha(v)))(\beta(u))$$

for all $v \in T_x X$ and $u \in E_x$. \square

From (2.1) it follows immediately that for any $v_1, v_2 \in T_x X$, we have

$$\eta_x(v_1) \circ \eta_x(v_2) = \eta_x(v_2) \circ \eta_x(v_1),$$

where η_x is constructed in (2.8). In view of this commutativity, there is a generalized eigenspace decomposition of E_x for $\{\eta_x(v)\}_{v \in T_x X}$. More precisely, we have distinct elements $u_1^x, \dots, u_m^x \in \Omega_{X,x}$ and a decomposition

$$E_x = \bigoplus_{i=1}^m E_x^i \quad (2.9)$$

such that

- for all $v \in T_x$ and $1 \leq i \leq m$,

$$\eta_x(v)(E_x^i) \subseteq E_x^i, \quad (2.10)$$

- the endomorphism of E_x^i

$$\eta_x(v)|_{E_x^i} - u_i^x(v) \cdot \text{Id}_{E_x^i} \quad (2.11)$$

is nilpotent.

Therefore, these elements $\{u_i^x\}_{i=1}^m$ are the joint generalized eigenvalues of $\{\eta_x(v)\}_{v \in T_x X}$. Note however that there is no ordering of the elements $\{u_i^x\}_{i=1}^m$. From Lemma 2.3 it follows immediately that the integer m is independent of x .

Let Y' denote the space of all pairs of the form (x, ϵ) , where $x \in X$ and

$$\epsilon : \{1, \dots, m\} \longrightarrow \{u_i^x\}_{i=1}^m$$

is a bijection. Clearly, Y' is an étale Galois cover of X with the permutations of $\{1, \dots, m\}$ as the Galois group. We note that Y' need not be connected. Fix a connected component $Y \subset Y'$. Let

$$\varpi : Y \longrightarrow X, \quad (x, \epsilon) \longmapsto x \quad (2.12)$$

be the projection. So ϖ is an étale Galois covering map.

For any $y = (x, \epsilon) \in Y$, and any $i \in \{1, \dots, m\}$, the element $\epsilon(i) \in \{u_i^x\}_{i=1}^m$ will be denoted by $\widehat{u}_i^{\varpi(y)}$.

Therefore, from (2.9) we have a decomposition

$$\varpi^* E = \bigoplus_{i=1}^m F_i, \quad (2.13)$$

where the subspace $(F_i)_y \subset (\varpi^* E)_y = E_{\varpi(y)}$, $y \in Y$, is the subspace of $E_{\varpi(y)}$ which is the generalized simultaneous eigenspace of $\{\eta_x(v)\}_{v \in T_{\varpi(y)} X}$ for the eigenvalue $\widehat{u}_i^{\varpi(y)}(v)$ (the element $\widehat{u}_i^{\varpi(y)}$ is defined above).

Clearly, (2.13) is a holomorphic decomposition of the holomorphic vector bundle $\varpi^* E$. Consider the Higgs field $\varpi^* \theta \in H^0(Y, \text{End}(\varpi^* E) \otimes \Omega_Y)$ on $\varpi^* E$, where $\Omega_Y = \varpi^* \Omega_X$ is the holomorphic cotangent bundle of Y . From (2.10) it follows immediately that

$$(\varpi^* \theta)(F_i) \subseteq F_i \otimes \Omega_Y. \quad (2.14)$$

Let

$$\theta_i := (\varpi^*\theta)|_{F_i} \quad (2.15)$$

be the Higgs field on F_i obtained by restricting $\varpi^*\theta$.

Equip Y with the pulled back Kähler form $\varpi^*\omega$. Consider the Hermitian structure ϖ^*h on ϖ^*E , where h , as before, is a Hermitian structure on E satisfying the Yang–Mills–Higgs equation for (E, θ) . It is straightforward to check that ϖ^*h satisfies the Yang–Mills–Higgs equation for $(\varpi^*E, \varpi^*\theta)$. In particular, $(\varpi^*E, \varpi^*\theta)$ is polystable. The restriction of ϖ^*h to the subbundle F_i in (2.13) will be denoted by h_i . Since

$$(\varpi^*E, \varpi^*\theta) = \bigoplus_{i=1}^m (F_i, \theta_i),$$

where θ_i is constructed in (2.15), and ϖ^*h satisfies the Yang–Mills–Higgs equation for $(\varpi^*E, \varpi^*\theta)$, it follows that h_i satisfies the Yang–Mills–Higgs equation for (F_i, θ_i) [Si1, p. 878, Theorem 1]. Consequently, (F_i, θ_i) is polystable. We note that the polystability of (F_i, θ_i) also follows from the fact that (F_i, θ_i) is a direct summand of the polystable Higgs vector bundle $(\varpi^*E, \varpi^*\theta)$.

Let

$$\mathrm{tr}(\theta_i) \in H^0(Y, \Omega_Y) \quad (2.16)$$

be the trace of θ_i . Let r_i be the rank of the vector bundle F_i . Define

$$\tilde{\theta}_i := \theta_i - \frac{1}{r_i} \mathrm{Id}_{F_i} \otimes \mathrm{tr}(\theta_i) \in H^0(Y, \mathrm{End}(F_i) \otimes \Omega_Y). \quad (2.17)$$

We note that $\tilde{\theta}_i$ is also a Higgs field on F_i .

Corollary 2.4. *The section $\theta_i \in H^0(Y, \mathrm{End}(F_i) \otimes \Omega_Y)$ in (2.15) is flat with respect to the connection on $\mathrm{End}(F_i) \otimes \Omega_Y$ constructed from h_i and $\varpi^*\omega$. Similarly, $\tilde{\theta}_i$ in (2.17) is flat with respect to this connection on $\mathrm{End}(F_i) \otimes \Omega_Y$.*

Proof. We noted earlier that h_i satisfies the Yang–Mills–Higgs equation for (F_i, θ_i) . From this it follows that h_i also satisfies the Yang–Mills–Higgs equation for $(F_i, \tilde{\theta}_i)$. Therefore, substitutions of (F_i, θ_i, h_i) and $(F_i, \tilde{\theta}_i, h_i)$ in place of (E, θ, h) in Proposition 2.2 yield the result. \square

Proposition 2.5. *The Higgs field $\tilde{\theta}_i$ on F_i in (2.17) vanishes identically.*

Proof. Since the endomorphism in (2.11) is nilpotent, it follows that

$$\tilde{\theta}_i(y)(v) \in \mathrm{End}(\varpi^*E_y) = \varpi^*\mathrm{End}(E_y) = \mathrm{End}(E_{\varpi(y)})$$

is nilpotent for all $y \in Y$ and $v \in T_y Y$. Consider the homomorphism

$$\tilde{\theta}_i : F_i \longrightarrow F_i \otimes \Omega_Y, \quad z \longmapsto \tilde{\theta}_i(y)(z) \quad \forall z \in (F_i)_y. \quad (2.18)$$

Let

$$\mathcal{V}_i := \text{kernel}(\tilde{\theta}_i) \subset F_i \quad (2.19)$$

be the kernel of it. From Corollary 2.4 it follows that the subsheaf $\mathcal{V}_i \subset F_i$ is a subbundle. We also note that \mathcal{V}_i is of positive rank.

Let

$$\tilde{\theta}_i^f = \tilde{\theta}_i \otimes \text{Id}_{\Omega_Y}$$

be the Higgs field on $F_i \otimes \Omega_Y$. Since $\varpi^*\omega$ is Kähler–Einstein, and h_i satisfies the Yang–Mills–Higgs equation for $(F_i, \tilde{\theta}_i)$, the Hermitian structure on $F_i \otimes \Omega_Y$ induced by the combination of h_i and $\varpi^*\omega$ satisfies the Yang–Mills–Higgs equation for $(F_i \otimes \Omega_Y, \tilde{\theta}_i^f)$. In particular, $(F_i \otimes \Omega_Y, \tilde{\theta}_i^f)$ is polystable.

Note that

$$\frac{\text{degree}(F_i \otimes \Omega_Y)}{\text{rank}(F_i \otimes \Omega_Y)} = \frac{\text{degree}(F_i)}{\text{rank}(F_i)} + \frac{\text{degree}(\Omega_Y)}{\text{rank}(\Omega_Y)} = \frac{\text{degree}(F_i)}{\text{rank}(F_i)}; \quad (2.20)$$

the last equality follows from the fact that $c_1(\Omega_Y) = 0$. The homomorphism $\tilde{\theta}_i$ in (2.18) is compatible with the Higgs fields $\tilde{\theta}_i$ and $\tilde{\theta}_i^f$ on F_i and $F_i \otimes \Omega_Y$ respectively, meaning $\tilde{\theta}_i \circ \tilde{\theta}_i = \tilde{\theta}_i \circ \tilde{\theta}_i$. From the definition of \mathcal{V}_i in (2.19) it follows immediately that $\tilde{\theta}_i|_{\mathcal{V}_i} = 0$. Hence $(\mathcal{V}_i, 0)$ is a Higgs subbundle of $(F_i, \tilde{\theta}_i)$. Since both $(F_i, \tilde{\theta}_i)$ and $(F_i \otimes \Omega_Y, \tilde{\theta}_i^f)$ are semistable of same slope (see (2.20)), we conclude that $(\mathcal{V}_i, 0)$ is a Higgs subbundle of $(F_i, \tilde{\theta}_i)$ of same slope (same as that of F_i). Now, as $(F_i, \tilde{\theta}_i)$ is polystable, the Higgs subbundle $(\mathcal{V}_i, 0)$ of same slope has a direct summand.

Let $(W_i, \theta_i^c) \subset (F_i, \tilde{\theta}_i)$ be a direct summand of $(\mathcal{V}_i, 0)$. If $W_i = 0$, then the proof is complete. So assume that $W_i \neq 0$.

Substituting (W_i, θ_i^c) in place of $(F_i, \tilde{\theta}_i)$ in the above argument and iterating the argument, we conclude that $\tilde{\theta}_i = 0$. \square

Corollary 2.6. *Let X be a compact 1-connected Calabi–Yau manifold. If (E, θ) is a polystable Higgs vector bundle on X , then $\theta = 0$.*

Proof. Since X is simply connected, it follows that ϖ in (2.12) is an isomorphism. We have $H^0(X, \Omega_X) = 0$, because $b_1(X) = 0$ and $\dim H^0(X, \Omega_X) = b_1(X)/2$. Therefore, $\text{tr}(\theta_i)$ in (2.16) vanishes identically, and hence $\tilde{\theta}_i$ in (2.17) is θ_i itself. Now Proposition 2.5 completes the proof. \square

3. INDEPENDENCE OF YANG–MILLS–HIGGS HERMITIAN STRUCTURE

As before, X is a compact connected Kähler manifold with $c_1(TX) = 0$, and ω is a Ricci-flat Kähler form on X . Let (E, θ) be a polystable Higgs vector bundle on X . Let h be a Hermitian structure on E satisfying the Yang–Mills–Higgs equation for (E, θ) . We will continue to use the set-up of Section 2.

Lemma 3.1. *The decomposition in (2.13) is orthogonal with respect to the pulled back Hermitian structure ϖ^*h on ϖ^*E .*

Proof. The decomposition in (2.13) gives a decomposition of the Higgs vector bundle $(\varpi^*E, \varpi^*\theta)$

$$(\varpi^*E, \varpi^*\theta) = \bigoplus_{i=1}^m (F_i, \theta_i),$$

where θ_i are constructed in (2.15). Recall that $(\varpi^*E, \varpi^*\theta)$ and all (F_i, θ_i) are polystable. If \tilde{h}_i , $1 \leq i \leq m$, is a Hermitian structure on F_i satisfying the Yang–Mills–Higgs equation for (F_i, θ_i) , then the Hermitian structure $\bigoplus_{i=1}^m \tilde{h}_i$ on ϖ^*E , constructed using the decomposition in (2.13), clearly satisfies the Yang–Mills–Higgs equation for $(\varpi^*E, \varpi^*\theta)$.

Any two Hermitian structures on ϖ^*E that satisfy the Yang–Mills–Higgs equation for $(\varpi^*E, \varpi^*\theta)$, differ by a holomorphic automorphism of the Higgs vector bundle $(\varpi^*E, \varpi^*\theta)$ [Si1, p. 878, Theorem 1]. In particular, there is a holomorphic automorphism

$$T : \varpi^*E \longrightarrow \varpi^*E$$

such that $(T \otimes \text{Id}_{\Omega_Y}) \circ (\varpi^*\theta) = (\varpi^*\theta) \circ T$, and

$$\bigoplus_{i=1}^m \tilde{h}_i(a, b) = \varpi^*h(T(a), T(b)). \quad (3.1)$$

Therefore, the lemma follows once it is shown that any holomorphic automorphism of the Higgs vector bundle $(\varpi^*E, \varpi^*\theta)$ preserves the decomposition in (2.13). Note that the decomposition in (2.13) is orthogonal for the above Hermitian structure $\bigoplus_{i=1}^m \tilde{h}_i$ on ϖ^*E . If the above automorphism T preserves the decomposition in (2.13), then from (3.1) it follows immediately that the decomposition in (2.13) is orthogonal with respect to ϖ^*h .

From the construction of the decomposition in (2.13) it follows that the m sections

$$\frac{1}{r_1} \text{tr}(\theta_1), \dots, \frac{1}{r_m} \text{tr}(\theta_m) \in H^0(Y, \Omega_Y)$$

in (2.16) and (2.17) are distinct; as mentioned just before (2.9), the elements $\{u_i^x\}_{i=1}^m$ are all distinct. Indeed, (2.13) is the generalized eigenspace decomposition for $\varpi^*\theta$, and $\frac{1}{r_1} \text{tr}(\theta_1), \dots, \frac{1}{r_m} \text{tr}(\theta_m)$ are the eigenvalues. It now follows that any automorphism of the Higgs vector bundle $(\varpi^*E, \varpi^*\theta)$ preserves the decomposition in (2.13). As observed earlier, this completes the proof. \square

Lemma 3.2. *The section*

$$\theta \bigwedge \theta^* \in C^\infty(X, \text{End}(E) \otimes \Omega_X^{1,1})$$

(see (2.2)) *vanishes identically.*

Proof. Consider θ_i defined in (2.15). From Proposition 2.5 it follows immediately that

$$\tilde{\theta}_i \bigwedge \tilde{\theta}_i^* = 0. \quad (3.2)$$

Since the decomposition in (2.13) is orthogonal by Lemma 3.1, from (3.2) and (2.17) we conclude that

$$(\varpi^* \theta) \bigwedge (\varpi^* \theta^*) = 0.$$

This implies that $\theta \wedge \theta^* = 0$. □

Theorem 3.3. *Let (E, θ) be a polystable Higgs vector bundle on X equipped with a Yang–Mills–Higgs structure h . Then h also satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle $(E, 0)$.*

Proof. In view of Lemma 3.2, this follows immediately from (2.2). □

Remark 3.4. It should be clarified that the converse of Theorem 3.3 is not valid. In other words, if h is an Hermitian structure on E satisfying the Yang–Mills–Higgs equation for $(E, 0)$, then h need not satisfy the Yang–Mills–Higgs equation for (E, θ) . The reason for it is that the automorphism group of $(E, 0)$ is in general bigger than the automorphism group of (E, θ) . To give an example, take X to be a complex elliptic curve equipped with a flat metric. Take E to be the trivial vector bundle $\mathcal{O}_X^{\oplus 2}$ on X of rank two. Let θ be the Higgs field on $\mathcal{O}_X^{\oplus 2}$ given by the matrix

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix};$$

fixing a trivialization of Ω_X , we identify the Higgs fields on $\mathcal{O}_X^{\oplus 2}$ with the 2×2 complex matrices. This Higgs vector bundle (E, θ) is polystable because the matrix A is semisimple. The Hermitian structure on $\mathcal{O}_X^{\oplus 2}$ given by the standard inner product on \mathbb{C}^2 satisfies the Yang–Mills–Higgs equation for $(E, 0)$, but this Hermitian structure does not satisfy Yang–Mills–Higgs equation for (E, θ) (because $AA^* \neq A^*A$).

Remark 3.5. Let (E, θ) be a polystable Higgs vector bundle on X . From Theorem 3.3 we know that the Higgs vector bundle $(E, 0)$ is polystable. Fix a Hermitian structure h_0 on E satisfying the Yang–Mills–Higgs equation for $(E, 0)$. Any other Hermitian structure on E that satisfies the Yang–Mills–Higgs equation for $(E, 0)$ differs from h_0 by a holomorphic automorphism of E . Take a holomorphic automorphism T of E such that the Hermitian structure $h := T^*h_0$ on E has the following property:

$$\theta \bigwedge \theta^{*h} = 0,$$

where θ^{*h} is the adjoint of θ constructed using h . From Lemma 3.2 and Theorem 3.3 it follows that such an automorphism T exists. The above Hermitian structure h satisfies the Yang–Mills–Higgs equation for (E, θ) .

4. POLYSTABLE PRINCIPAL HIGGS G -BUNDLES

Let G be a connected reductive affine algebraic group defined over \mathbb{C} . The Lie algebra of G will be denoted by \mathfrak{g} . As before, X is a compact connected Kähler manifold equipped with a Ricci-flat Kähler form ω . Let $E_G \rightarrow X$ be a holomorphic principal G -bundle. Its adjoint vector bundle $E_G \times^G \mathfrak{g}$ will be denoted by $\text{ad}(E_G)$. A Higgs field on E_G is a holomorphic section

$$\theta \in H^0(X, \text{ad}(E_G) \otimes \Omega_X)$$

such that the section $\theta \wedge \theta$ of $\text{ad}(E_G) \otimes \Omega_X^2$ vanishes identically. A Higgs G -bundle on X is a pair of the form (E_G, θ) , where E_G is a holomorphic principal G -bundle on X , and θ is a Higgs field on E_G .

Fix a maximal compact subgroup

$$K_G \subset G.$$

A Hermitian structure on a holomorphic principal G -bundle E_G on X is a C^∞ reduction of structure group of E_G

$$E_{K_G} \subset E_G$$

to the subgroup K_G . There is a unique C^∞ connection ∇ on the principal K_G -bundle E_{K_G} such that the connection on E_G induced by ∇ is compatible with the holomorphic structure of E_G [At, p. 191–192, Proposition 5]. Using the decomposition $\mathfrak{g} = \text{Lie}(K) \oplus \mathfrak{p}$, given any Higgs field θ on E_G , we have

$$\theta^* \in C^\infty(X; \text{ad}(E_G) \otimes \Omega_X^{0,1}).$$

Let (E_G, θ) be a Higgs G -bundle on X . The center of the Lie algebra \mathfrak{g} will be denoted by $z(\mathfrak{g})$. Since the adjoint action of G on $z(\mathfrak{g})$ is trivial, we have an injective homomorphism

$$\psi : X \times z(\mathfrak{g}) \hookrightarrow \text{ad}(E_G) \tag{4.1}$$

from the trivial vector bundle with fiber $z(\mathfrak{g})$. This homomorphism ψ produces an injective homomorphism

$$\widehat{\psi} : z(\mathfrak{g}) \hookrightarrow H^0(X, \text{ad}(E_G)).$$

A Hermitian structure $E_{K_G} \subset E_G$ is said to satisfy the Yang–Mills–Higgs equation for (E_G, θ) if there is an element $c \in z(\mathfrak{g})$ such that

$$\Lambda_\omega(\mathcal{K}(\nabla) + \theta \bigwedge \theta^*) = \widehat{\psi}(c),$$

where $\mathcal{K}(\nabla)$ is the curvature of the connection ∇ associated to the reduction E_{K_G} , and θ^* is defined above.

It is known that (E_G, θ) admits a Yang–Mills–Higgs Hermitian structure if and only if (E_G, θ) is polystable [Si2], [BS, p. 554, Theorem 4.6]. (See [BS] for the definition of a polystable Higgs G -bundle.)

Lemma 4.1. *Let (E_G, θ) be a Higgs G -bundle on X equipped with an Hermitian structure $E_{K_G} \subset E_G$ that satisfies the Yang–Mills–Higgs equation for (E_G, θ) . Then*

$$\theta \bigwedge \theta^* = 0.$$

Proof. This follows by applying Lemma 3.2 to the Higgs vector bundle associated to (E_G, θ) for the adjoint action of G on \mathfrak{g} . Consider the adjoint Higgs vector bundle $(\text{ad}(E_G), \text{ad}(\theta))$. The reduction E_{K_G} produces a Hermitian structure on the vector bundle $\text{ad}(E_G)$ that satisfies the Yang–Mills–Higgs equation for $(\text{ad}(E_G), \text{ad}(\theta))$. Now Lemma 3.2 says that

$$\text{ad}(\theta) \bigwedge \text{ad}(\theta)^* = 0.$$

This immediately implies that the C^∞ section $\theta \bigwedge \theta^*$ of $\text{ad}(E_G) \otimes \Omega_X^{1,1}$ is actually a section of $\psi(z(\mathfrak{g})) \otimes \Omega_X^{1,1}$, where ψ is the homomorphism in (4.1).

Take any holomorphic character $\chi : G \rightarrow \mathbb{C}^*$. Let

$$L^\chi := E_G \times^\chi \mathbb{C} \rightarrow X$$

be the holomorphic line bundle associated to E_G for χ . The Higgs field θ defines a Higgs field on L^χ using the homomorphism of Lie algebras

$$d\chi : \mathfrak{g} \rightarrow \mathbb{C} \tag{4.2}$$

associated to χ ; this Higgs field on L^χ will be denoted by θ^χ . Since L^χ is a line bundle, we have $\theta^\chi \bigwedge (\theta^\chi)^* = 0$ (Lemma 3.2 is not needed for this). As $\theta \bigwedge \theta^*$ is a section of $\psi(z(\mathfrak{g})) \otimes \Omega_X^{1,1}$, from this it can be deduced that $\theta \bigwedge \theta^* = 0$. Indeed, given any nonzero element $v \in z(\mathfrak{g})$, there is a holomorphic character

$$\chi : G \rightarrow \mathbb{C}^*$$

such that $d\chi(v) \neq 0$ (defined in (4.2)). □

Theorem 4.2. *Let (E_G, θ) be a polystable Higgs G -bundle on X , and let $E_{K_G} \subset E_G$ be an Hermitian structure that satisfies the Yang–Mills–Higgs equation for (E_G, θ) . Then the Hermitian structure $E_{K_G} \subset E_G$ also satisfies the Yang–Mills–Higgs equation for $(E_G, 0)$.*

Proof. In view of the Yang–Mills–Higgs equation for (E_G, θ) , this follows immediately from Lemma 4.1. □

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